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TOWARD AN ENGINEERING THEORY FOR ADHESIVE JOINTS

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ABSTRACT

The development of high strength structural adhesives has created many new uses for adhesive joints and established a need for additional theoretical and experimental study. An adhesive lap joint is analyzed taking advantage of its similarity to a sandwich structure for which an established theory exists. The principle of minimum complementary energy is used to develop the governing equations, which are equivalent to a single differential equation of eighth order. Included in the complementary energy expression are contributions due to shear and normal stresses in the adhesive layer and due to shear, axial, bending, and normal stresses in the adherend.

CHAPTER I

INTRODUCTION

Adhesives have been in use for hundreds of years. Genghis Khan's remarkable success was partially due to his archers' use of small powerful bows made by laminating different materials with animal and vegetable derivatives used as gluing agents [1, 2].*
Exposure to moisture severely weakened these bows and caused them to fail after some use. Typical of all adhesives until recently, loss of strength due to moisture strictly limited the structural importance of such materials until the development of synthetics.

With the development of synthetic resin adhesives came the possibility of new applications. Previously only wooden components were bonded, but now metal-to-metal joints between dissimilar materials became feasible.

There are a number of reasons why the replacement of mechanical fasteners and welded joints by adhesives is attractive. The continuous nature of the joint reduces stress concentrations and provides longer fatigue life [3, 4]. Dissimilar materials can be effectively joined with adhesives where an alternative joining procedure would have been nonexistent or prohibitively difficult before. A good example is the metal-paper honeycomb sandwich. New structural applications such as honeycomb sandwich and fibrous composites have been made possible with

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* Numbers in square brackets refer to references, page .

strong adhesives. Yet another important advantage is an economic one. Adhesive joining eliminates a number of production costs associated with setting up and assembling joints with rivets and spot welds [1]. The lap joint is particularly favorable in this respect because of its basic simplicity and ease of fabrication.

The inability of a stress analyst or designer to determine whether or not he has correctly analyzed the stress distribution in an actual adhesive joint is the most formidable disadvantage to the use of this joining procedure. There are several reasons why this is so.

A realistic adhesive joint is influenced by factors that analytical methods usually neglect. A statistical evaluation of experimental results is required to predict accurately the regions in which failure will occur. Air bubbles or voids, residual stresses due to improper curing, variation of the adhesive layer thickness, and the "notch effect" due to convex or concave adhesive layer boundary all significantly influence the stress concentrations and the stress distributions within an adhesive joint [5-7]. Variation of thickness is so important that some authors consider thickness to be the essential criteria for strength predictions [8, 9]. This means that the care taken in the fabrication of an adhesive joint is a determining factor of its strength--a factor which, unfortunately, cannot be accounted for in a deterministic fashion.

Theoretical efforts have left important areas unexplored or inadequately examined. There is no unified view that spans the entire range of useful parameters, although many analyses exist which treat limited cases. There are a number of theories available that are based on simplifications of the actual three dimensional elastic problem. Unfortunately, their application is limited by the fact that it is unclear in which instances the simplifications are valid.

Experimental research and widespread engineering use of adhesives is hampered by the lack of a fully reliable nondestructive test. Where experimental research is held back analytical work is also hindered. As Sherlock Holmes [16] said, "I have no data yet. It is a capital mistake to theorize before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts." Theory without experimental confirmation cannot be completely relied upon.

These three facts--the statistical nature of adhesive joints, uncertainty surrounding available theories, and the lack of reliable nondestructive testing techniques--have limited the use of adhesives and plagued joint designers. Nevertheless, analytical and semi-empirical results currently must be used as the best available means of estimating joint characteristics for design purposes. After manufacturing has commenced, destructive tests on a sampling basis are normally conducted to insure joint reliability.

The analysis of the complete three-dimensional adhesive joint problem is difficult and, in view of the uncertainty of practical fabrication techniques, cannot be justified. Fortunately, investigators have established some degree of correlation between experimental data and their particular simplified theories. Reference [1] provides a good survey of the contributions of Volkersen, de Bruyne, Goland and Reissner, Plantema, Mylonas, and others [7, 11-14].

The present thesis is concerned with establishing a unified theory that spans the unexplored area between a rigorous elastic analysis [13] and the highly simplified theory of Lunsford [10]. Initial simplifications due to similarity of adhesive joints and sandwich structure make possible a straight forward analysis accounting for what are believed to be the most important factors influencing joint behavior.

A complementary energy expression is obtained and combined by means of Lagrange multipliers with the equilibrium equations. The principle of minimum complementary energy is employed to obtain equations of consistent deformation which can be reduced to a single eighth order equation.

CHAPTER II

ANALYSIS

Formulation of the Problem

In any engineering theory it is desirable to simplify the actual physical problem in order to reduce the difficulty of obtaining numerical results, particularly when the neglected effects are of minor importance. Nevertheless, it is advantageous to retain as many contributing factors as is necessary to adequately describe the problem. The analysis of adhesive joints should be approached in this spirit. This problem is an intricate one, but it is highly academic to account for all details because of the uncertainties and irregularities introduced by practical fabrication techniques.

The simplest adhesive joint is the lap joint. Since it is the most commonly used in actual applications of adhesive joints, it is considered here. Furthermore the analysis will be restricted to a lap joint which has one adherend much stiffer than the other. (A "rigid" idealization will be used.) The remaining adherend thickness t will be assumed to be much greater than the adhesive layer thickness h so that the ratio of h to t will be negligibly small. An inverse relation exists between joint strength and adhesive thickness; therefore a very thin adhesive layer can be expected.

Adhesive lap joints resemble ordinary sandwich structures, such as those with high strength faces and thick cores. A sandwich core is loaded primarily in shear with the effect of face parallel normal stresses of secondary importance; the same is true of the adhesive layer. Axial and bending stresses are the primary loading of both sandwich face sheets and adherends, but transverse normal and shear effects must be included for the latter. From the established theory of sandwich structures, the assumption of a state of anti-plane stress in the core is used for the adhesive layer [15].

The analysis will be conducted by obtaining the equilibrium equations of the upper, non-rigid adherend. An expression for the complementary energy of the system will be derived including effects of shear and normal stresses in the adhesive layer and shear, axial, bending, and normal stresses of the adherend. The equilibrium equations, multiplied by Lagrange multipliers, will be added to the complementary energy expression. The principle of minimum complementary energy is used along with the variational calculus to obtain the governing equations, which are reduced to a single eighth order differential equation.

Adhesive Layer

It is assumed that a state of anti-plane stress exists and that there are no body forces or residual stresses acting on the adhesive layer. Attention is restricted to a unit width in the y direction.*

The equilibrium equations are

$$\sigma_{xx,x} + \tau_{xy,y} + \tau_{xz,z} = 0 \qquad (1)$$

$$\tau_{yx,x} + \sigma_{yy,y} + \tau_{yz,z} = 0 \qquad (2)$$

* See Figure 1 for coordinates.

$$\tau_{zx,x} + \tau_{zy,y} + \sigma_{zz,z} = 0 \tag{3}$$

The usual notation for stresses is used with σ's denoting normal stresses and τ's denoting shear stresses.

The above assumptions are equivalent to

$$\sigma_{xx} = \tau_{xy} = \tau_{yx} = \sigma_{yy} = \tau_{yz} = \tau_{zy} = 0 \tag{4}$$

The simplified equations are

$$\bar{\tau}_{,z} = 0 \tag{5}$$

$$\bar{\tau}_{,x} + \sigma_{,z} = 0 \tag{6}$$

where

$$\bar{\tau} = \tau_{zx} \quad , \quad \sigma = \sigma_{zz} \tag{7}$$

Equation (5) indicates that $\bar{\tau}$ is a function of x alone.

$$\bar{\tau} = \bar{\tau}(x) \tag{8}$$

Equation (6) indicates

$$\sigma = S - z\,\bar{\tau}_{,x} \tag{9}$$

where S is a function of x only.

$$S = S(x) \tag{10}$$

Adherend

The arrangement, dimensions, and coordinates of the adhesive joint to be considered are shown in Figure 1. Figure 2 illustrates a typical element of the upper adherend, from which the following equilibrium equations can be deduced by inspection.

$$N_{,x} = \bar{\tau} \tag{11}$$

$$Q_{,x} = S - \frac{h}{2} \bar{\tau}_{,x} \tag{12}$$

$$M_{,x} = Q - \frac{t}{2} \bar{\tau} \tag{13}$$

The Energy Expression

The derivation of the complementary energy expression can be found in Appendix A. The contribution of the adhesive layer shear and normal stresses is

$$U_c = \frac{1}{2} \int_o^{\ell} \left[\frac{S^2 h}{E_a} + \frac{(\bar{\tau})^2 h}{G_a} + \frac{(\bar{\tau})^2 h^3}{12E_a}\right] dx \tag{14}$$

The contribution of the adherend shear, axial, bending, and normal stresses is

$$U_c = \frac{1}{2} \int_o^{\ell} \left\{\left[\frac{N^2}{Et} + \frac{12M^2}{Et^3}\right] - \frac{\nu}{E}\left[\frac{NS}{2} - \frac{6}{5}\frac{MS}{t} - \left(\frac{3h}{t} + 1\right)\frac{t}{12} N\bar{\tau}_{,x}\right.\right. \tag{15}$$

$$\left.\left. + \left(\frac{6h}{t} + 1\right) \frac{M\bar{\tau}_{,x}}{10}\right] + \left[\frac{13}{35}\frac{t}{E} S^2 - \frac{t^2}{105E}\left(\frac{39h}{t} + 11\right) S\bar{\tau}_{,x}\right.$$

$$+ \frac{t^3}{420E}\left(\frac{39h^2}{t^2} + \frac{22h}{t} + 4\right)\left(\bar{\tau}_{,x}\right)^2\Bigg] + \left[\frac{6}{5}\,\frac{Q^2}{Gt} + \frac{t}{G}(\bar{\tau})^2 - \frac{1}{5}\,\frac{Q\bar{\tau}}{G}\right]\Bigg\}\ dx$$

By adding (14) and (15) and the sum of the product of equations (11-13) and Lagrange multipliers, the desired expression is obtained.

$$U_c = \int_0^{\ell} \left\{\frac{1}{2}\left[\frac{N^2}{Et} + \frac{12M^2}{Et^2} - \frac{\nu}{E}\,\frac{NS}{2} + \frac{6}{5}\,\frac{\nu}{E}\,\frac{MS}{t} + \frac{\nu t}{12E}\left(\frac{3h}{t} + 1\right) N\bar{\tau}_{,x}\right.\right. \tag{16}$$

$$-\left(\frac{6h}{t} + 1\right)\frac{M\bar{\tau}_{,x}}{10} + \frac{13}{35}\,\frac{t}{E}\,S^2 - \frac{t}{105E}\left(\frac{39h}{t} + 11\right)S\bar{\tau}_{,x} + \frac{t^3}{420E}\left(\frac{39h^2}{t^2} + \frac{22h}{t} + 4\right)\left(\bar{\tau}_{,x}\right)^2$$

$$+ \frac{6Q^2}{5Gt} + \frac{2}{15}\,\frac{t}{G}(\bar{\tau})^2 - \frac{1}{5}\,\frac{Q\bar{\tau}}{G} + \frac{S^2h}{E_a} + \frac{h}{G_a}(\bar{\tau})^2 + \frac{(\tau)^2h^3}{12E_a}\Bigg]$$

$$+ \left[L_1\left(N_{,x} - \bar{\tau}\right) + L_2\left(Q_{,x} - S + \frac{h}{2}\,\bar{\tau}_{,x}\right) + L_3\left(Q - \frac{t}{2}\,\bar{\tau} - M_{,x}\right)\right]\Bigg\}\ dx$$

L_1, L_2, and L_3 are Lagrange multipliers that physically correspond to the displacement parallel to the joint, the displacement normal to the joint, and the adherend bending rotation, respectively. The following dimensionless variables and parameter are now introduced:

$$m = \frac{12M}{Et} \qquad n = \frac{N}{Et} \qquad q = \frac{Q}{Et} \qquad \tau = \frac{\bar{\tau}}{E} \tag{17}$$

$$s = \frac{S}{E} \qquad x = \frac{X}{\mu t} \qquad \lambda_1 = \frac{L_1}{t} \qquad \lambda_2 = \frac{L_2}{t}$$

$$\lambda_3 = L_3 \qquad K = \frac{Eh}{E_a t}$$

By rewriting (16) and neglecting $\frac{h}{t}$ because it is small, the expression becomes

$$U_c = Et\mu \int_o^{\ell} \frac{1}{2} \Big[n^2 + \frac{m^2}{12} + \left(\frac{13}{35} + K\right)s^2 + \frac{6}{5}\frac{E}{G}q^2$$

$$+ \left(\frac{2}{15}\frac{E}{G} + K\frac{E_a}{G_a}\right)\tau^2 + \frac{(\tau_{,x})^2}{105\mu} - \left(\frac{11}{105}s - \frac{\nu}{12}n + \frac{\nu}{120}m\right)\frac{\tau_{,x}}{\mu}$$

$$- \frac{1}{5}\frac{E}{G}q\tau - \frac{\nu}{2}ns + \frac{\nu}{10}ms\Big] + \Big[\lambda_1\left(\frac{n_{,x}}{\mu} - \tau\right)$$

$$+ \lambda_2\left(\frac{q_{,x}}{\mu} - s + \frac{h}{2t}\tau_{,x}\right) + \lambda_3\left(q - \frac{\tau}{2} - \frac{m_{,x}}{12\mu}\right)\Big] dx$$

Governing Equations

The principle of minimum complementary energy states

$$\delta U_c = 0 \tag{19}$$

δn, δm, δs, and $\delta\tau$ are independent variations; the corresponding Euler Equations are:

$$\delta n: \quad n - \frac{\nu}{24}\frac{\tau_{,x}}{\mu} - \frac{\nu}{4}s - \frac{\lambda_{1,x}}{\mu} = 0 \tag{20}$$

$$\delta m: \quad \frac{m}{12} - \frac{\nu}{240}\frac{\tau_{,x}}{\mu} + \frac{\nu}{20}s + \frac{\lambda_{3,x}}{12\mu} = 0 \tag{21}$$

$$\delta s: \quad \left(\frac{13}{35} + K\right)s - \frac{11}{210}\frac{\tau_{,x}}{\mu} - \frac{\nu}{4}n + \frac{\nu}{20}m - \lambda_2 = 0 \tag{22}$$

$$\delta q: \quad \frac{6}{5}\frac{E}{G}q - \frac{1}{10}\frac{E}{G}\tau - \frac{\lambda_{2,x}}{\mu} + \lambda_3 = 0 \tag{23}$$

$$\delta\tau: \quad \left(\frac{2}{15}\frac{E}{G} + \frac{E_a}{G_a}\right)\tau - \frac{\tau_{,xx}}{\mu^2} + \frac{11}{210}\frac{s_{,x}}{\mu} - \frac{\nu}{24}\frac{n_{,x}}{\mu} + \frac{\nu}{240}\frac{m_{,x}}{\mu} \tag{24}$$

$$- \frac{1}{10}\frac{E}{G}q - \lambda_1 - \frac{h}{2t}\frac{\lambda_{2,x}}{\mu} - \frac{\lambda_3}{2} = 0$$

Equations (20) - (24) are shown in Appendix B to be equivalent to the following eighth order governing differential equation:

$$\left(\frac{1}{12} + K\right)\frac{D^8\tau}{\mu^8} - \left[\frac{69}{20}\nu + \frac{53}{10}\frac{E}{G} + \left(14\frac{E}{G} + 39\frac{E_a}{G_a} - \frac{21\nu}{4} + 105\frac{E_a}{G_a}K\right)K\right]\frac{D^6\tau}{\mu^6} \tag{25}$$

$$+ \left[102 + \frac{63}{4}\left(\frac{E}{G}\right)^2 - \frac{63}{2}\nu^2 - \frac{21}{2}\frac{E}{G}\nu + \left(420 + 126\frac{E}{G}\frac{E_a}{G_a} - 126\frac{E_a}{G_a}\nu\right)K\right]\frac{D^4\tau}{\mu^4}$$

$$- \left(189\nu + 546\frac{E}{G} + 1260\frac{E_a}{G_a}K\right)\frac{D^2\tau}{\mu^2} + 1260\tau = 0$$

The differential equation has eight associated boundary conditions, four of which must be satisfied at each end of the adhesive joint. The usual boundary value problem for joints are those for which the horizontal force, vertical force, bending moment, and adhesive shear stress assume prescribed values at the joint edges.

CHAPTER III

DISCUSSION

The eighth order differential equation (25) is the governing equation for an adhesive lap joint. It is valid only for lap joints in which one adherend is much stiffer than the other. Furthermore, the flexible adherend thickness must be much greater than the adhesive layer thickness. This equation is more general than the idealized problem it represents would suggest. Adhesive layers must be thin, many applications involve attachment of flexible materials to a relatively rigid structure, and lap joints are quite common.

A comparison of the governing equation with previous theories is possible. The energy expression for the adherend, equation (15), has four bracketted quantities. They are, respectively, axial and bending energy, coupling term between axial and bending energy and normal stress energy, energy of normal stresses, and shear stress contribution. If the adherend shear contribution is neglected, the resulting equation would lead to Goland and Reissner's results [13]. If adherend shear and normal stresses are neglected, then the equations would reduce to Plantema's results [14]. Lunsford's theory is obtained if shear, normal, and bending stress contributions of the adherend are neglected along the normal stress terms of the adhesive layer energy expression, equation (14) [10].

The theory presented cannot be considered complete until experimental confirmation is obtained. Experimental research at the Georgia Institute of Technology Aerospace Structures Laboratory is in progress, and it is hoped an assessment of the theory's usefulness will be forthcoming.

CHAPTER IV

CONCLUSIONS

A unified engineering theory for adhesive lap joints has been developed using a modified version of the principle of minimum complementary energy. Attention is restricted to joints with one adherend assumed rigid and to joints whose adhesive material is idealized as linearly elastic. Simplified stress distributions are determined from elementary considerations and are used to construct the appropriate energy functional. Energy contributions included are shear and normal stresses in the adhesive layer and shear, axial, bending, and normal stresses in the adherend.

The governing equations are reduced to a single eighth order equation. The theories of previous investigators are obtainable from the present theory when appropriate energy contributions are neglected. Currently, no experimental confirmation of the theory's validity is available. However, research on adhesive joints at the Georgia Institute of Technology Aerospace Structures Laboratory is being conducted and an evaluation of the theory is planned.

APPENDICES

APPENDIX A

DERIVATION OF COMPLEMENTARY ENERGY EXPRESSION

The complementary energy of the adhesive layer is given by

$$U_c = \frac{1}{2}\int_0^{\ell}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\frac{(\bar{\tau})^2}{G_a} + \frac{\sigma^2}{E_a}\right] dz\ dx \tag{A1}$$

Equation (A1) is integrated over the thickness of the layer after substitution from equation (9).

The following expression is obtained:

$$U_c = \frac{1}{2}\int_0^{\ell}\left[\frac{S^2 h}{E_a} + \frac{(\bar{\tau})^2 h}{G_a} + \frac{(\bar{\tau})^2 h^3}{12E_a}\right] dx \tag{A2}$$

The adherend strip is considered to be in a state of plane strain ($\epsilon_{yy} = 0$). The energy expression including axial, bending, normal, and shear effects is

$$U_c = \int_0^{\ell}\int_{-\frac{t}{2}}^{\frac{t}{2}}\left(\frac{\sigma_{xx}^{\ 2}}{E} - \frac{\nu}{E}\sigma_{xx}\sigma_{zz} + \frac{\sigma_{zz}^{\ 2}}{E} + \frac{\sigma_{xz}^{\ 2}}{G}\right) dz\ dx \tag{A3}$$

where

$$E = \frac{E'}{1 - \nu'^2} \quad , \quad \nu = \frac{2\nu'}{1 - \nu'} \tag{A4}$$

in which E' is Young's Modulus and ν' is Poisson's Ratio.

σ_{xx} is given by

$$\sigma_{xx} = \frac{N}{t} + \frac{12}{t} Mz \tag{A5}$$

and the energy contribution by

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{\sigma_{xx}^{2}}{E} \, dz = \frac{N^2}{Et} + \frac{12M^2}{Et^3} \tag{A6}$$

By using the following equation:

$$\sigma_{xx,x} + \sigma_{xz,z} = 0$$

and by substituting from equations (10) and (12), an expression can be obtained for σ_{xz} upon satisfaction of the boundary equations.

$$\sigma_{xz} = \frac{6Q}{t}\left(\frac{1}{4} - \frac{z^2}{t^2}\right) + \bar{\tau}\left(\frac{3z^2}{t^2} - \frac{z}{t} - \frac{1}{4}\right) \tag{A7}$$

The energy contribution can now be obtained.

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{\sigma_{xz}^{2}}{G} \, dz = \frac{6Q^2}{5Gt} + \frac{2}{15}\frac{(\bar{\tau})^2 t}{G} - \frac{1}{5}\frac{Q\bar{\tau}}{G} \tag{A8}$$

An expression for σ_{zz} can be obtained from the following equation:

$$\sigma_{xz,x} + \sigma_{zz,z} = 0 \tag{A9}$$

Equations (A7) and (A9) are used to evaluate σ_{zz}, which becomes upon satisfaction of boundary conditions:

$$\sigma_{zz} = \left(\frac{1}{2} - \frac{3}{2t} z + \frac{2}{t^3} z^3\right) Q_{,x} + \left(- \frac{t}{8} + \frac{z^3}{t^2} + \frac{z^2}{2t} + \frac{z}{4}\right) \overline{\tau}_{,x} \tag{A10}$$

The energy contributions can now be evaluated and are found to be

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{\sigma_{zz}{}^2}{E} \, dz = \left[\frac{13}{35} \frac{t}{E} S^2 - \frac{S\overline{\tau}_{,x}}{105} \frac{t^2}{E} \left(39\frac{h}{t} + 11\right)\right. \tag{A11}$$

$$\left. + \frac{(\overline{\tau}_{,x})^2}{420} \frac{t^3}{E} \left(\frac{39h^2}{t^2} + 22\frac{h}{t} + 4\right)\right]$$

and

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{\nu}{E} \sigma_{xx} \sigma_{zz} \, dz = \frac{\nu}{E} \left[\frac{NS}{2} - \frac{6}{5} \frac{MS}{t} - \left(\frac{3h}{t} + 1\right) \frac{t}{12} N\overline{\tau}_{,x}\right. \tag{A12}$$

$$\left. + \left(\frac{6h}{t} + 1\right) \frac{M\overline{\tau}_{,x}}{10}\right]$$

The adherend energy contributions are combined according to equation (A3) and the result is as follows:

$$U_c = \frac{1}{2} \int_0^{\ell} \left\{\left[\frac{N^2}{Et} + \frac{12M^2}{Et^3}\right] - \frac{\nu}{E}\left[\frac{NS}{2} - \frac{6}{5} \frac{MS}{t} - \left(\frac{3h}{t} + 1\right)\frac{t}{12} N\overline{\tau}_{,x}\right.\right. \tag{A13}$$

$$\left. + \left(\frac{6h}{t} + 1\right) \frac{M\overline{\tau}_{,x}}{10}\right] + \left[\frac{13}{35} \frac{t}{E} S^2 - \frac{t^2}{105E} \left(\frac{39h}{t} + 11\right) S\overline{\tau}_{,x}\right.$$

$$\left.\left. + \frac{t^3}{420E}\left(\frac{39h^2}{t^2} + \frac{22h}{t} + 4\right)\left(\overline{\tau}_{,x}\right)^2\right] + \left[\frac{6}{5} \frac{Q^2}{Gt} + \frac{2}{15} \frac{t}{G}(\overline{\tau})^2 - \frac{1}{5} \frac{Q\overline{\tau}}{G}\right]\right\} dx$$

Respectively, the square bracketted quantities are axial and bending energy, coupling between axial and bending energy and normal energy, normal stress energy, and energy due to shear stresses.

APPENDIX B

REDUCTION OF INDEPENDENT GOVERNING EQUATIONS TO A SINGLE EIGHTH ORDER GOVERNING EQUATION

In addition to the five independent variations mentioned previously in equations (20-24), there are three remaining, $\delta\lambda_1$, $\delta\lambda_2$, and $\delta\lambda_3$. The equilibrium equations in nondimensional form are obtained by setting these three independent variations equal to zero.

$$\frac{n_{,x}}{\mu} = \tau \qquad \text{(B1)}$$

$$s = \frac{q_{,x}}{\mu} + \frac{h}{2t}\frac{\tau_{,x}}{\mu} \qquad \text{(B2)}$$

$$q = \frac{\tau}{2} + \frac{m_{,x}}{12\mu} \qquad \text{(B3)}$$

Using equation (B3), equation (B2) becomes $(\frac{h}{t} \ll 1)$

$$s = \frac{1}{2}\frac{\tau_{,x}}{\mu} + \frac{m_{,xx}}{12\mu^2} \qquad \text{(B4)}$$

λ_2 can be solved for in equation (22), λ_3 can be solved for in equation (23), and then λ_1 can be obtained from (24). These values can be substituted into equations (20) and (21) to obtain

$$n - \frac{\nu}{24}\frac{\tau_{,x}}{\mu} - \frac{\nu s}{4} - \left(\frac{1}{12}\frac{E}{G} + \frac{E_a}{G_a}K\right) - \frac{\tau_{,xxx}}{60\mu^3} \tag{B5}$$

$$+ \left(\frac{2}{15} + \frac{K}{2}\right)\frac{s_{,xx}}{\mu^2} - \frac{\nu}{12}\frac{n_{,xx}}{\mu^2} + \frac{\nu}{48}\frac{m_{,xx}}{\mu^2} - \frac{1}{2}\frac{E}{G}\frac{q_{,x}}{\mu} = 0$$

$$\frac{m}{12} - \frac{\nu}{240}\frac{\tau_{,x}}{\mu} + \frac{\nu s}{20} + \frac{1}{12}\left(\frac{13}{35} + K\right)\frac{s_{,xx}}{\mu^2} - \frac{11}{2520}\frac{\tau_{,xxx}}{\mu^3} \tag{B6}$$

$$- \frac{\nu}{48}\frac{n_{,xx}}{\mu^2} + \frac{\nu}{240}\frac{m_{,xx}}{\mu^2} + \frac{1}{120}\frac{E}{G}\frac{\tau_{,x}}{\mu} - \frac{1}{10}\frac{E}{G}q_{,x} = 0$$

Upon rearrangement and substitution of (B1) and (B4), equations (B5) and (B6) become

$$\frac{6}{5}\frac{E}{G}\frac{q_{,x}}{\mu} - \left[m + \frac{\nu}{10}\frac{m_{,xx}}{\mu^2} + \frac{1}{12}\left(\frac{13}{35} + K\right)\frac{m_{,xxxx}}{\mu^4}\right] = \tag{B7}$$

$$\frac{1}{10}\frac{E}{G}\frac{\tau_{,x}}{\mu} + \left(\frac{2}{15} + \frac{K}{2}\right)\frac{\tau_{,xxx}}{\mu^3}$$

$$6\frac{E}{G}\frac{q_{,xx}}{\mu^2} - \left(\frac{2}{15} + \frac{K}{2}\right)\frac{m_{,xxxxx}}{\mu^5} = 12\tau \tag{B8}$$

$$- \left(3\nu + \frac{E}{G} + 12\frac{E_a}{G_a}K\right)\frac{\tau_{,xx}}{\mu^2} + \left(\frac{3}{5} + 3K\right)\frac{\tau_{,xxxx}}{\mu^4}$$

Writing (B7) and (B8) in operator notation, they become

$$\left[\frac{6}{5}\,\frac{E}{G}\,\frac{D}{\mu}\right]q - \left[1 + \frac{\nu}{10}\,\frac{D^2}{\mu^2} + \frac{1}{12}\left(\frac{13}{35} + K\right)\frac{D^4}{\mu^4}\right]m = \left[\frac{1}{10}\,\frac{E}{G}\,\frac{D}{\mu} + \left(\frac{2}{15} + \frac{K}{2}\right)\frac{D^3}{\mu^3}\right]\tau \tag{B9}$$

$$\left[6\,\frac{E}{G}\,\frac{D^2}{\mu^2}\right]q - \left[\left(\frac{2}{15} + \frac{K}{2}\right)\frac{D^5}{\mu^5}\right]m = \tag{B10}$$

$$\left[12 - \left(3\nu + \frac{E}{G} + 12\,\frac{E_a}{G_a}\,K\right)\frac{D^2}{\mu^2} + \left(\frac{3}{5} + 3K\right)\frac{D^4}{\mu^4}\right]\tau$$

where D is the differential operator $\frac{d(\;)}{dx}$. Equations (B9) and (B10) may be solved for q and m in terms of τ.

$$q = \frac{P\tau}{L} \tag{B11}$$

$$m = \frac{R\tau}{L} \tag{B12}$$

where L, P, and R are differential operators defined below.

$$\left.\begin{aligned}
P &= \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} \\[2ex]
P_{11} &= 12 - \left(3\nu + \frac{E}{G} + 12\,\frac{E_a}{G_a}\,K\right)\frac{D^2}{\mu_2} + \left(\frac{3}{5} + 3K\right)\frac{D^4}{\mu^4} \\[2ex]
P_{12} &= -\left(\frac{2}{15} + \frac{K}{2}\right)\frac{D^5}{\mu^5} \\[2ex]
P_{21} &= \frac{1}{10}\,\frac{E}{G}\,\frac{D}{\mu} + \left(\frac{2}{15} + \frac{\nu}{2}\right)\frac{D^3}{\mu^3}
\end{aligned}\right\} \tag{B13}$$

$$
\left.
\begin{aligned}
&P_{22} = -\left[1 + \frac{\nu}{10}\frac{D^2}{\mu^2} + \frac{1}{12}\left(\frac{13}{35} + K\right)\frac{D^4}{\mu^4}\right] \\
&\qquad\qquad R = \begin{vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{vmatrix} \\
&R_{11} = \frac{6E}{G}\frac{D^2}{\mu^2} \\
&R_{12} = 12 - \left(3\nu + \frac{E}{G} + 12\,\frac{E_a}{G_a}\,K\right)\frac{D^2}{\mu^2} + \left(\frac{3}{5} + 3K\right)\frac{D^4}{\mu^4} \\
&R_{21} = \frac{6}{5}\frac{E}{G}\frac{D}{\mu} \\
&R_{22} = \frac{1}{10}\frac{E}{G}\frac{D}{\mu} + \left(\frac{2}{15} + \frac{K}{2}\right)\frac{D^3}{\mu^3}
\end{aligned}
\right\} \quad (B14)
$$

$$
\left.
\begin{aligned}
&\qquad\qquad L = \begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} \\
&L_{11} = R_{11},\ L_{12} = P_{12},\ L_{21} = R_{21},\ L_{22} = R_{22}
\end{aligned}
\right\} \quad (B15)
$$

If the operator L is applied to equation (B3), the following result is obtained:

$$
Lq - \frac{L}{2}\,\tau - \frac{D}{12\mu}\,Lm = 0 \qquad (B16)
$$

The terms q and m can now be eliminated with the aid of (B11) and (B12); the result is

$$\left(P - \frac{L}{2} - \frac{D}{12\mu} R\right) \tau = 0 \tag{B17}$$

If the differential operators are expanded, the above equation can be written in the form (25).

FIGURES

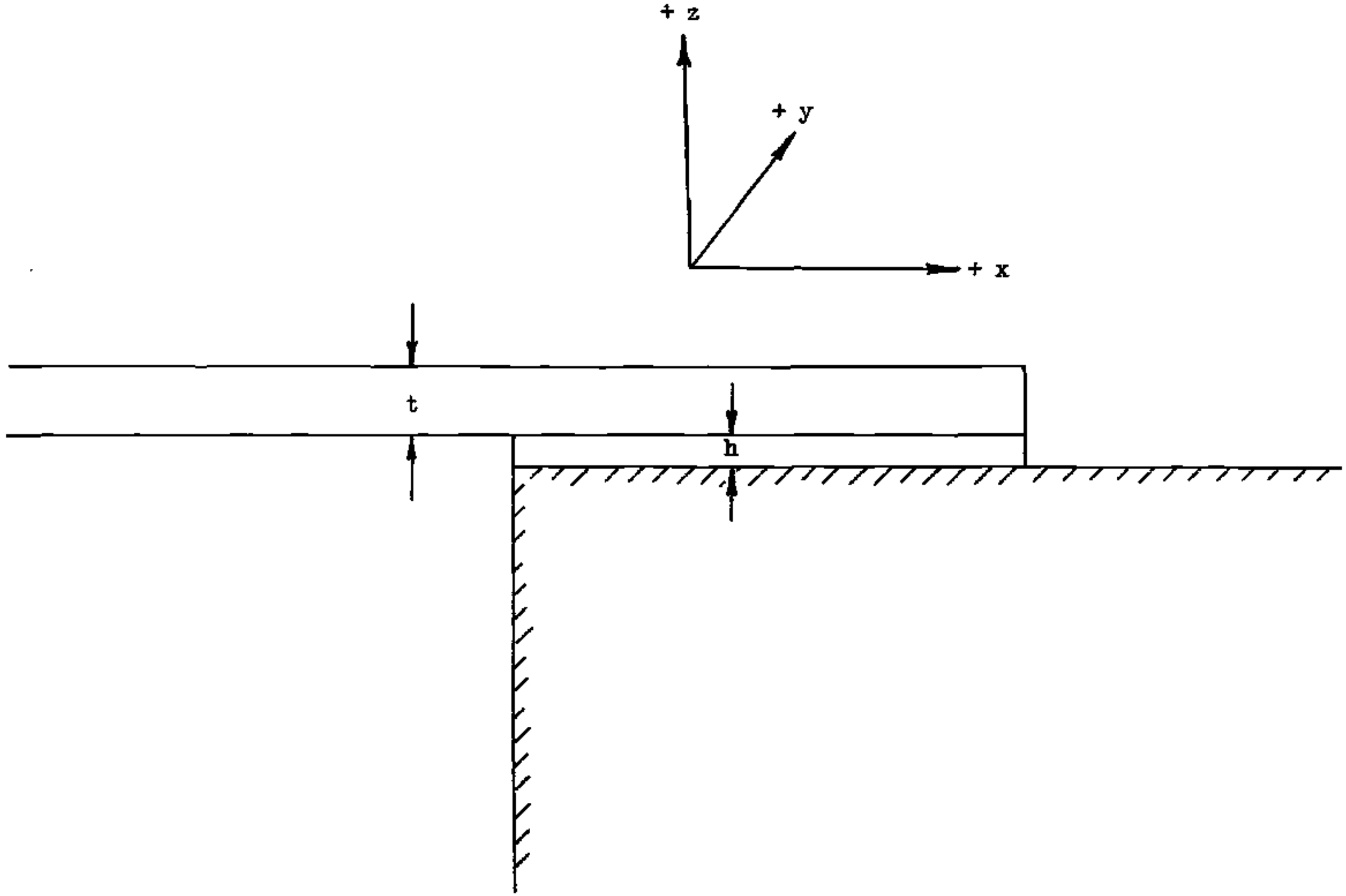

Figure 1. Adhesive Lap Joint.

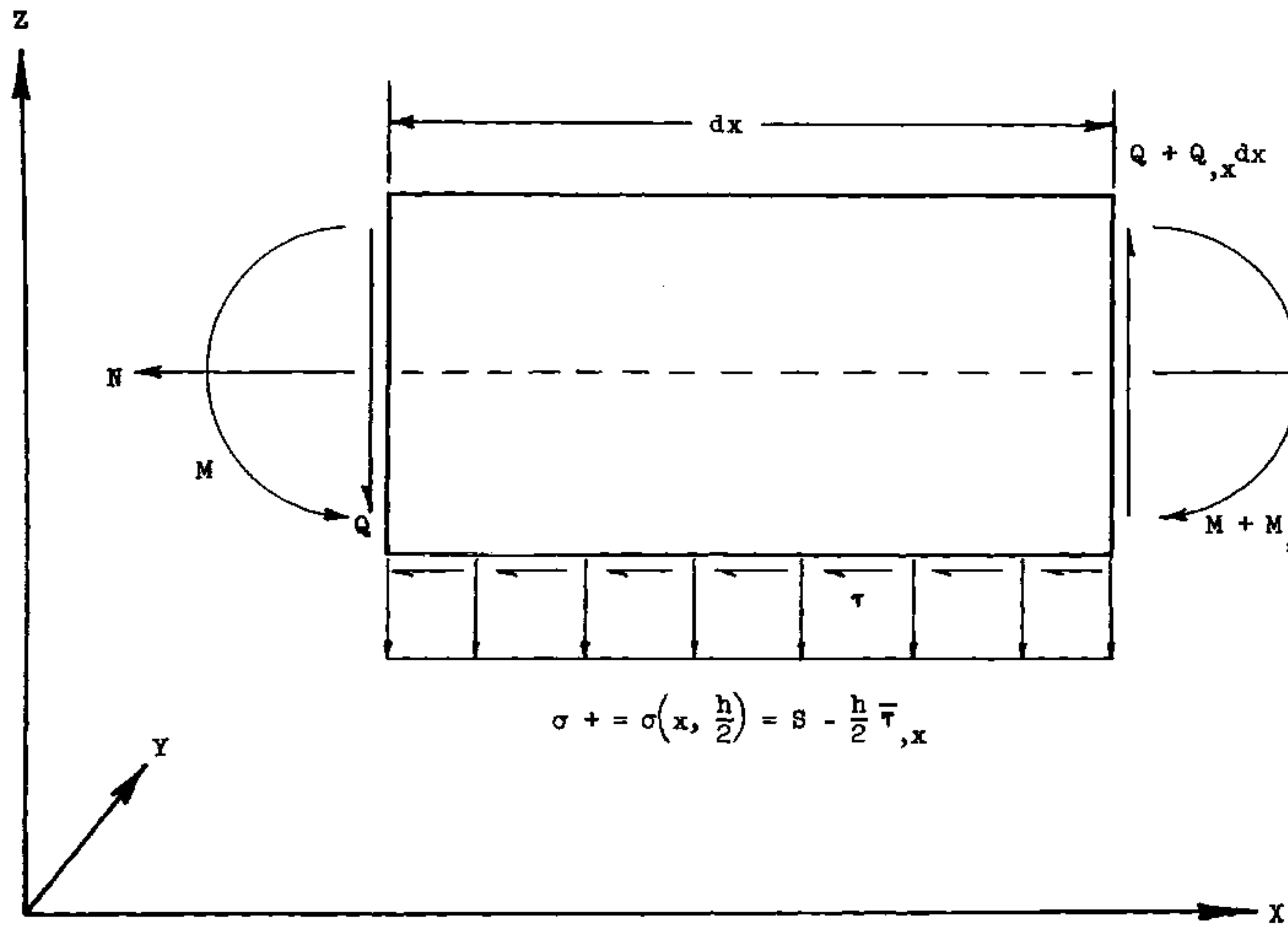

Figure 2. Adherend Element.

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